Degree incremental solutions of the generalized cauchy-riemann system

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Abstract. This article is devoted to finding the solution space of the class of exponentially increasing functions that do not belong to the class of multipliers of the equation of generalized analytic functions, for example, in the case of constant or linear functions.

Keywords: Generalized analytic function, Liouville's theorem, polynomial
**Introduction:** The Generalized Cauchy-Riemann system is a fundamental concept in complex analysis, which deals with functions of a complex variable. The system consists of two partial differential equations that determine whether a given function is holomorphic, i.e., differentiable at every point in its domain. The solutions to the Generalized Cauchy-Riemann system play a crucial role in understanding the behavior of complex functions and their properties.

Degree incremental solutions of the Generalized Cauchy-Riemann system refer to a specific class of solutions that are obtained by incrementing the degree of the functions involved. This approach allows for a more systematic and structured analysis of the solutions, leading to a deeper understanding of the underlying mathematical principles.

This article deals with «Increasing power solutions of the generalized Cauchy-Riemann system».

Academician I. N. Vegua [1] $L_{p,1}(E), p > 2$ proved the famous Liouville's theorem in the theorem of analytic functions for generalized analytic functions in the class of multipliers.

This article $L_{p,1}(E), p > 2$ is devoted to finding the solution space of the class of exponentially increasing functions that do not belong to the class of multipliers of the equation of generalized analytic functions, for example, in the case of constant or linear functions.

*E* in the plane, if $A(z), B(z) \in L_{p,2}(E)$, then

$$
\frac{\partial w}{\partial \bar{z}} + A(z)w + B(z)\bar{w} = 0
$$

(1)

used notation for regular solutions of the system of equations. $U_{p,2}(E)$ Here is given in $L_{p,2}(E)$ the entire plane by [2] 

And

$$
f(z) \in L_p(|z| \leq 1), |z|^{-\nu} f \left( \frac{1}{z} \right) \in L_p(|z| \leq 1), p \geq 1, \nu \in \mathbb{R}
$$

satisfying the conditions $f(z)$ is defined. If $w(z)$ there is a generalized solution from the class $z = \infty$ of equation (1),
then it is called a generalized analytic function, and if
\( U_{p,2}(E), p > 2 \) is its only pole, then it is called a generalized
polynomial.

The following theorem generalizing Liouville's theorem [3] is proved: if (1) has multiples of the system \( A(z), B(z) \in L_{p,2}(E) \), then \( z = \infty \) in the vicinity of the infinite distance
point

\[
|w| \leq C|z|^N
\]  

(2)

(1) \( w(z) \in U_{p,2}(E) \) is a generalized solution of the rank class
that is a regular solution of the system on the entire plane
satisfying the estimate \( N_{U_{p,2}(E)}, p > 2 \)

We study here the multipliers of this problem
\( A(z) = a\bar{z}, B(z) = b z \). Here, we assume that \( a \) and \( b \) are real numbers

\[
z = z_1 e^{i\phi}, w = w_1 e^{i\psi}
\]

by using the transformations and \( \phi \) and \( \psi \) choosing

\[
\frac{\partial w_1}{\partial \bar{z}_1} + |a| z_1 w_1 + |b| z_1 \bar{w}_1 = 0, |w_1| \leq C|z_1|^N
\]

can be brought to the form

(1) solution of the system

\[
w(z) = \varphi(z)e^{iy(z)} + \psi(z)e^{ix(z)}
\]  

(3)

can be expressed in the form, where \( \varphi, \psi \) are analytic and
\( y, x \) are real continuous functions. Then from condition (2).

\[
w(z) = e^{iy(z)} P(z) + e^{ix(z)} Q(z)
\]

let's express it. Here are polynomials of \( P(z), Q(z) \) any
\( N \) degree. We express them by the sum of homogeneous
polynomials:

\[
P = \sum_{n=0}^{N} P_n, Q = \sum_{n=0}^{N} Q_n, P_n = \sum_{k=0}^{n} a_{k,n-k} z^k \bar{z}^{n-k}, Q_n = \sum_{k=0}^{n} b_{k,n-k} z^k \bar{z}^{n-k}
\]

(4)
This is for defining polynomials

\[ e^{i\gamma}p_z + (a\bar{z} + i\gamma_z)e^{i\chi}p + bze^{-i\gamma}\bar{Q} + e^{i\gamma}Q_z + (a\bar{z} + i\chi_z)e^{i\chi}Q + bze^{-i\chi}\bar{Q} = 0 \]

system and from that from scratch there is a different polynomial and only if

\[ \gamma(z) = -\chi(z) \]

when only I see it. Then \( P, Q \) to determine the polynomials

\[ P_z + (az - i\chi_z)P + b\bar{z}Q = 0, \quad Q_z + (a\bar{z} + i\chi_z)Q + b\bar{z}\bar{P} = 0 \quad (5) \]

systems we will take. And this of systems from scratch different having solutions for

\[ \chi(z) + az^2 + \bar{a}\bar{z}^2 + \beta\bar{z}z \alpha \in C, \beta \in R \quad (6) \]

second in order to be a homogeneous polynomial necessary and enough.

Expressions (4), (6) and system (5). and the same degree compare and \( P, Q \) match multiplication operations polynomials determine three in

\[ [(a - 2\bar{a}i)\bar{z} - i\beta z]P_N + bz\bar{Q}_N = 0, Q_N[(a + 2\alpha i)\bar{z} + ibz] + \bar{P}_Nbz = 0 \quad (7_N) \]

\[ [(a - 2\bar{a}i)\bar{z} - i\beta z]P_{N-1} + bz\bar{Q}_{N-1} = 0, Q_{N-1}[(a + 2\alpha i)\bar{z} + ibz] + \bar{P}_{N-1}bz = 0 \quad (7_{N-1}) \]

\[ \frac{\partial P_N}{\partial \bar{z}} + [(a - 2\bar{a}i)\bar{z} - i\beta z]P_{N-2} + bz\bar{Q}_{N-2} = 0, \frac{\partial Q_N}{\partial \bar{z}} + Q_{N-2}[(a + 2\alpha i)\bar{z} + ibz] + bzh\bar{P}_{N-2} = 0 \quad (7_{N-2}) \]

\[ \frac{\partial P_2}{\partial \bar{z}} + [(a - 2\bar{a}i)\bar{z} - i\beta z]P_0 + bz\bar{Q}_0 = 0 \quad (7_0) \]

\[ \frac{\partial Q_2}{\partial \bar{z}} + Q_0[(a + 2\alpha i)\bar{z} + ibz] + bzh\bar{P}_0 = 0 \]
\[
\frac{\partial P_1}{\partial z} = 0, \quad \frac{\partial Q_1}{\partial \bar{z}} = 0 \quad (7)
\]

equations system in we will take.

(7 \_N \_N) [4] equations system from scratch different solve
i m i there is there and only then, if their determinant

\[
\Delta = \begin{vmatrix}
(a - 2ia\bar{a}) - i\beta z & bz \\
(b\bar{a}) & (a - 2a) - i\beta \bar{z}
\end{vmatrix} = 0
\]

if , that is

\[(ai\beta - 2a\beta)z^2 + (ai\beta - 2a\beta)\bar{z}^2 + (a^2 - 2a\bar{a} - 2ia\bar{a} - 4|\alpha|^2 - \beta^2 - b^2)z\bar{z} = 0\]

if From the equality \(\beta = 0\), of \(2a\beta - 2a\beta = 0\) and \(2a\beta - 2a\beta = 0\)
and from the equality \(-2ai(a + \bar{a}) = 0\) of \((a + \bar{a}) = 0\), we find \(\alpha \neq 0\)
that \(\alpha\) is a pure imaginary number, \(a^2 - 4|\alpha|^2 - \beta^2 = 0\) find from
Eq \(\alpha = \pm i\sqrt{a^2 - b^2}\)

\[
\chi(z) = \frac{i}{2}\sqrt{a^2 - b^2}(\bar{z}^2 - z^2)
\quad (8)
\]

we will determine.

If \(\Delta \neq 0\) so, then from the equations (7 \_N \_N), (7 \_N-\_1)
\(P_N \equiv 0, Q_N \equiv 0, P_{N-1} \equiv 0, Q_{N-1} \equiv 0\). Thus (7 \_N-\_2), (7 \_N-\_3), etc. All from
Eqs

\[
P_k \equiv 0, Q_k \equiv 0, k = 0,7, ..., N
\]

we find that so, \(\Delta \neq 0\) our problem has only zero solution, and
from the expression of the solution of the problem (1),
(2) \(a < b\), then always \(\Delta \neq 0\).

Now \(a = b\) let's look at the situation. Then, \(z_1 = az\) with the
help of the transformation (1), (2) problem

\[
\frac{\partial w}{\partial z} + \bar{z}w + z\bar{w} = 0, |w| < C|z|^N
\quad (9)
\]

and (7) systems with their compatibility conditions

\[
zP_N + z\bar{P}_N = 0, \text{Im} \frac{\partial P_N}{\partial \bar{z}} = 0, zP_{N-1} + z\bar{P}_{N-1} = 0, \text{Im} \frac{\partial P_{N-1}}{\partial \bar{z}} = 0
\]
Consider a system of equations, again all even-numbered

\[ P_{2m}(z) = 0, n = 2m \]

Therefore, in system (14), \( n = 2k - 1 \) we consider only odd-numbered systems:

\[ \bar{z}P_{2k-1} + z\bar{P}_{2k-1} = -\frac{\partial P_{2k+1}}{\partial z}, \text{Im} \frac{\partial P_{2k+1}}{\partial z} = 0, \]

That is, each \( P_{2k+1}(z) \) polynomial depends on only one specific parameter.

Therefore, in system (14), \( n = 2k - 1 \) we consider only odd-numbered systems:
required compatibility \( Im \frac{\partial P_{2k+1}}{\partial z} = 0 \), condition

\[
zP_{2k-1} = -\frac{1}{2} \frac{\partial P_{2k+1}}{\partial z} + i(\bar{z}Q_{2k-1})
\]  

(16)

Find and integrate it

\[
Q_{2k-1}(z) = TIm \frac{\partial}{\partial z} \left\{ \frac{1}{2} \frac{\partial P_{2k+1}}{\partial z} \right\} + \rho_{2k-1}z(z^2 - \bar{z}^2)^{k-1}
\]  

(17)

We find that here \( T \) is an integral operator, and \( \rho_{2k-1} \) actual setting

\[
\rho_{2k-1} + (-1)^k \bar{\rho}_{2k-1} = 0, k = 3, 4, ..., m 2m + 1 = N
\]  

(18)

Satisfies the condition. Now from the last three (10) equations

\[
P_1(z) = \tau z, P_3(z) = (Re \tau)z(z^2 - \bar{z}^2), P_0 = 0
\]  

(19)

So, \( a = b \) in [5], we see from formulas (12), (19) that the problem \( [\frac{N-1}{2}] + 1 \) has linearly independent solutions. We understand linear independence in the field of real numbers. Here, \( [\frac{N-1}{2}] \) the \( \frac{N-1}{2} \) whole part of the number is marked.

We only \( a > b \) have to study the situation. In this case, the solution of problem (1), (2).

\[
w(z) = P e^{\frac{\sqrt{a^2 - b^2}}{2}(z^2 - \bar{z}^2)} + Q e^{\frac{\sqrt{a^2 - b^2}}{2}(\bar{z}^2 - z^2)}
\]

is searched in the form, where \( P(z), Q(z) \) - polynomials of the form (4).

Egyp\( v = a + \sqrt{a^2 - b^2}, \mu = a - \sqrt{a^2 - b^2} \) if we enter the symbols, then \( P_n, Q_n \) to identify polycoms

\[
vzP_N + b\bar{z}Q_N = 0, \mu \bar{z}Q_N + b\bar{z}P_N = 0 \quad (20_1)
\]

\[
vzP_{N-1} + b\bar{z}Q_{N-1} = 0, \mu \bar{z}Q_{N-1} + b\bar{z}P_{N-1} = 0 \quad (20_2)
\]
\[ \frac{\partial P_n}{\partial z} + v\bar{z}P_{n-2} + bz\bar{Q}_{n-2} = 0, \quad \frac{\partial Q_n}{\partial z} + \mu\bar{z}Q_{n-2} + bz\bar{P}_{n-2} = 0, \quad n = 3, \ldots, N \quad (20_n) \]

\[ \frac{\partial P_{n-1}}{\partial z} + v\bar{z}P_{n-3} + bz\bar{Q}_{n-3} = 0, \quad \frac{\partial Q_{n-1}}{\partial z} + \mu\bar{z}Q_{n-3} + bz\bar{P}_{n-3} = 0, \quad n = 3, \ldots, N \quad (20_{n-1}) \]

systems. And from system (20).

\[ \bar{z}P_N = -\frac{b}{v}\bar{z}\bar{Q}_N \quad (21) \]

and from (20\text{ }) system

\[ \frac{\partial P_N}{\partial z} + v\bar{z}P_{N-2} + bz\bar{Q}_{N-2} = 0, \quad \frac{\partial Q_N}{\partial z} + \mu\bar{z}Q_{N-2} + bz\bar{P}_{N-2} = 0 \quad (22) \]

we get and multiply the second of these by \( b \)

\[ \frac{v}{b} \frac{\partial Q_N}{\partial z} = -bz\bar{Q}_{N-2} - v\bar{z}P_{N-2} \]

we get the equation. Then (22) is based on the first equation of the system

\[ \frac{\partial P_N}{\partial z} = \frac{v}{b} \frac{\partial Q_N}{\partial z} \quad (23) \]

we get the equality. Now differentiate by the equation (20_1) \( z \)

\[ v\bar{z}\frac{\partial P_N}{\partial z} + b\bar{Q}_N + bz\frac{\partial Q_N}{\partial z} = 0 \quad (24) \]

take the equation, multiply it by \( -z \) and then use formulas (22) and (23) \( P_N \) to determine the polynomial

\[ v\bar{z}^2 \frac{\partial P_N}{\partial z} + \mu z^2 \frac{\partial P_N}{\partial z} = v\bar{z}P_N \quad (25) \]

we come to the equation. Integrate this \( N \) for all pairs \( P_{2k} = 0 \), \( (26) \).
for all thrones $N = 2k + 1$

$$P_{2k+1}(z) = \rho_{2k+1}z(\mu z^2 - vz^2)^k$$

(27)

we find that

And (21) from the system $\bar{Q}_N(z) = -\frac{v}{b} \bar{P}_N$ so,

$$Q_{2k+1}(z) = -\frac{v}{b} \bar{P}_{2k+1}z(\mu z^2 - vz^2)^k, Q_{2k} \equiv 0$$

(28)

Now $20_n$, proceeding from the system from top to bottom, based on formulas (26)-(28), we can easily determine that all polynomials of even degree are equal to zero, and $N$ to determine $20_{n-1}$ polynomials of degree less than $20_n$ according to which one is of odd degree), ( $20_n$) system

$$vzP_k + bz\bar{Q}_k = -\frac{\partial p_{k+2}}{\partial z}, \mu zQ_k + b\bar{P}_k = -\frac{\partial Q_{k+2}}{\partial z}$$

(29)

written in the form, the polynomials here satisfy the system $P_{k+2}, Q_{k+2}$ ( $20_1$), i.e

$$vzP_{k+2} + bz\bar{Q}_{k+2}, \mu zQ_{k+2} + b\bar{P}_{k+2} = 0$$

(30)

And (30) systems are differentiated through and

$$\frac{\partial}{\partial z}(bz\bar{Q}_k) = vz \frac{\partial P_k}{\partial z} - \frac{b}{\mu} \frac{\partial^2 \bar{Q}_{k+2}}{\partial z^2}$$

taking into account that

$$bz \frac{\partial \bar{Q}_k}{\partial z} = \mu z \frac{\partial P_k}{\partial z}$$

(31)

we get the equality Moving to the node, and then $v\mu = b^2$ using the equation (29) from the second of the system

$$bz\bar{Q}_k = -vzP_k - \frac{b}{\mu} \frac{\partial Q_{k+2}}{\partial z}$$

(32)

we get the equality
to \( z(29) \).

\[
v \dd{}{z} P_k + \frac{\partial}{\partial z}(b z \dd{\overline{Q}_k}{z}) = -\dd{2}{z^2} P_{k+2}
\]

Putting equalities (31), (32) into the equation, by multiplying the system obtained from it

\[
v z \dd{}{z} P_k + \mu z^2 \dd{P_k}{z} - v \dd{P_k}{z} = g(z, \overline{z}), k = 0, 1, 2, ..., N - 2, \quad (33)
\]

\( P_k(z) \) we get the equation for defining the polynomial here

\[
g(z, \overline{z}) = \frac{b}{\mu} \dd{Q_{k+2}}{z} - z \dd{2}{z^2} P_{k+2}
\]

known polynomial. And now

\[
\frac{dz}{v \dd{\overline{z}}{z}} = \frac{d \dd{\overline{z}}{z}}{\mu z^2} = \frac{dP_k}{v \dd{P_k}{z} + g(z, \overline{z})}
\]

one of the first integrals of the system

\[
\mu z^2 - v \dd{z}{z}^2 = C_1
\]

Therefore, there \( P_k \) to determine the polynomial

\[
\frac{d \dd{\overline{z}}{z}}{C_1 + v \dd{\overline{z}}{z}^2} = \frac{dP_k}{v \dd{P_k}{z} + \dd{\overline{g}}{C_1, \overline{z}}}
\]

differential equation or

\[
\frac{dP_k}{d \dd{\overline{z}}{z}} = \frac{v \dd{z}}{C_1 + v \dd{\overline{z}}{z}^2} P_k + f(\overline{z}, C_1)
\]

we get the equation, were

\[
f(\overline{z}, C_1) = \frac{\dd{\overline{g}}(C_1, \overline{z})}{C_1 + v \dd{\overline{z}}{z}^2} \dd{\overline{g}}(C_1, \overline{z}) = g \left( \frac{(C_1 + v \dd{\overline{z}}{z})}{\mu}, \overline{z} \right)
\]
(34) Integrating the equation

\[ C_2 = \frac{P_k}{\sqrt{\mu z}} - F(\bar{z}, C_1) \]

we will find.

Then in general solve i m

\[ \phi(C_1, C_2) = 0 \text{ or } \phi(\mu z^2 - v\bar{z}^2, \frac{P_k}{\sqrt{\mu z}} - F(\bar{z}, \mu z^2 - v\bar{z}^2)) = 0 \]

Today

\[ P_k(z) = \sqrt{\mu z}(\phi(\mu z^2 - v\bar{z}^2) + F(\bar{z}, \mu z^2 - v\bar{z}^2)) \]

here \( F \) the first function of the function by \( f(z, C_1) \) \( \phi(\mu z^2 - v\bar{z}^2) \) through \( \mu z^2 - v\bar{z}^2 \) from the expression dependent polynomial marked.

So, \( P \) - odd degree polynomial:

\[ P_{2k+1} = \rho_{2k+1}z(\mu z^2 - vz)^k + F \]

and \( Q_{2k+1}(z) \) the polynomial (29) is from the second term of equations will be found. Then every step Each time, a complex parameter appears sits. So, \( a > b \) when Calculations (1) - (2) are accurate numbers in the field \( N + 1 \) there are linearly independent solutions.

In conclusion, we proved the following theorem.

Theorem. If yes \( a < b \), then problem (1) - (2) has only zero solution, if \( a = b \) yes, then problem (1) - (2) \( N + 1 \) has linearly independent solution, and if if \( a > b \), then problem (1) - (2) \( 2N + 2 \) is linearly independent there is a solution.

We understand linear independence in the field of real numbers

Conclusion

The famous Liouville theorem in the theory of analytic functions states that a generalized analytic function of increasing power in the entire plane is a polynomial. And for generalized analytic functions, I. N. Vekua showed that this theorem holds \( L_p(E), p > 2 \) even if the multipliers are from the
class. For example, when the multipliers are constant, they $L_{p,2}(E), p > 2$ do not belong to the class. In this last case, V.S. Vinogradov discovered spaces of increasing degree. In this article, we have shown the space of increasing power even when multipliers are linear functions (such functions do not belong to the class of I.N. Vequa).

In conclusion, degree incremental solutions of the Generalized Cauchy-Riemann system offer a valuable approach to studying complex functions and their properties. By incrementing the degree of the functions, we can systematically analyze the solutions and gain a deeper understanding of their structure and behavior. This incremental approach can lead to new insights, results, and applications in various fields, making it an important topic in complex analysis.

References: