Mathematical modeling of typical electrical circuits, oscillators and pendulums

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Abstract. The work is devoted to mathematical and simulation modeling of electromechanical systems, in which oscillations in electrical circuits, oscillators and pendulums are studied under the action of forces of various physical nature. To compile the equations of electromechanical systems, the apparatus of analytical mechanics was used, in which the electromagnetic and mechanical quantities characterizing the system appear as formally equal. The equations of motion are based on the Lagrange-Maxwell equations. The simulation was performed using computer algebra programs.

Keywords: Electromechanical systems Lagrange-Maxwell equations phase trajectories computer algebra programs.
Electromechanical systems are called systems in which mechanical and electromagnetic processes are essentially interconnected. In mechanics, the characteristics of a state are generalized coordinates and velocities (or impulses). For electromechanical systems, they constitute the first group of characteristics, the second includes quantities that describe electromagnetic processes.

Electromechanical systems (EMS) are widely used in many areas of technology. Examples of electromechanical systems: electric motors, generators, non-contact (electromagnetic, electrostatic) solid body suspensions, electrical measuring instruments. The development of EMC goes both along the path of improving technical means and in the direction of searching for new control algorithms. For rational design and analysis of the properties of such systems, modern engineering practice requires the creation of correct mathematical models, which must contain differential equations of mechanical motion, as well as equations of electromagnetic processes. For compiling the equations of electromechanical systems, the apparatus of analytical mechanics is very convenient, in which the electromagnetic and mechanical quantities characterizing the system appear as formally equal in rights and the equations of motion are derived using the Lagrangian formalism.

1 Series ferroresonant LRC circuit with current dependent inductance.

Consider the response of a series LRC circuit to a DC input voltage (Fig. 1). This refers to a ferroresonant circuit with an inductance that depends on the current [1].

The circuit equation can be written as

\[ n \frac{dq}{dt} + Ri + \frac{q}{c} = E, \quad i = \frac{dq}{dt}, \]  

(1)
where \( \phi \) – the magnetic flux of the core, \( n \) – the number of turns of the core winding, \( q \) – the charge of the capacitor, \( R \) – the ohmic resistance, \( i \) – the current, \( C \) – the capacity of the capacitor, \( E \) – the electromotive force (EMF), \( t \) – time. In this case, we mean a ferroresonant circuit with an inductance that depends on the current.

Assume that the magnetization curve of the core is determined by the equation

\[
\phi = c_1 \cdot n \cdot i + c_2 \cdot \text{th}(n \cdot i),
\]

(2)

where \( c_1 \) and \( c_2 \) – constant, dependent on the material of the core.

By entering the numerical values of the parameters

\[ n = 1, R = 0.2, C = 2.5, c_1 = 0.08, c_2 = 0.4 \]

and using equations (1) and (2), we obtain

\[
\left(1.2 - \text{th}^2 \frac{dq}{dt}\right) \frac{d^2 q}{dt^2} + 0.5 \frac{dq}{dt} + q = 2.5E.
\]

The results of modeling in the form of dependences of charge and current on time, as well as the phase portrait are presented in fig. 2.
You can get a number of solutions if you change any parameter, for example – $E$ (Fig. 3).

![Figure 3](image_url)

**Figure 3**

Dependence of $q(t)$ on parameter $E$

It is also possible to construct a parametric surface, for example, the dependence of $q(t)$ on the initial conditions $q(0)$ and time $t$ (Fig. 4):

![Figure 4](image_url)

**Figure 4**

Dependence of $q(t)$ on initial conditions $q(0)$ and time $t$

### 2. Oscillator Ueda

Consider the nonlinear Ueda oscillator [2] under periodic external influence. In the presence of an external periodic
influence, terms containing a clear dependence on time appear in the oscillation equations, and the nonlinear oscillator turns into a non-autonomous system that can demonstrate complex dynamics and the transition to chaos. If we assume that the nonlinear dependence of the restoring force for the oscillator has the form \( f(x) \), and the friction force is proportional to the speed, then we arrive at the equation

\[
\ddot{x} + \gamma \dot{x} + f(x) = a \sin \omega t,
\]

where \( \gamma \) - the dissipation parameter, \( a \) and \( \omega \) set the frequency and amplitude of influence.

Another option is parametric excitation, when we have a periodic dependence of the coefficient in the equation on time:

\[
\ddot{x} + \gamma \dot{x} + (1 + a \sin \omega t) f(x) = 0.
\]

Examples can be a pendulum with a thread length that changes periodically and a nonlinear LCR circuit in which the capacitance or inductance changes periodically in time.

Both with force and parametric excitation of a nonlinear dissipative oscillator, the parameters plane usually presents regions of different periodic and chaotic regimes, period doubling bifurcations, folds, assemblies and "crossroads".

An example can be a system of type (3) with a cubic nonlinear function \( f(x) = x^3 \). It is called the Duffing oscillator or the Wedy oscillator. The mechanical system described by the Duffing equation with external periodic influence is presented in Fig. 5 [2]. The ball is fixed on a vertically installed elastic plate, and the coefficient of elasticity is selected so that at small deflection angles, the turning force of the elasticity exactly compensates the deflecting moment of gravity. The Uedi oscillator can be implemented in the same way as an electrical circuit - an oscillating circuit with a nonlinear inductance.
Figures 6–8 show graphs simulating the solution of equation (3) at $f(x) = x^3$. It is accepted here: $a = 2$, $\gamma = 0.1$, $\nu = 0.5$, $\nu = \frac{dx}{dt}$.

- **Figure 5**: Oscillator Ueda
- **Figure 6**: Dependency coordinates $x$ from time

- **Figure 7**: Addiction of speed $v$ as a function of time
Simulations with variable parameters can be performed (Fig. 9-11). Let's take the parameters as variables $a = (0.5, 1.0, 1.5, 2.0)$ and $\nu = (0.1, 0.2, 0.3, 0.4, 0.5)$:

<table>
<thead>
<tr>
<th>$a$</th>
<th>$x_1(t)$</th>
<th>$x_2(t)$</th>
<th>$x_3(t)$</th>
<th>$x_4(t)$</th>
<th>$x_5(t)$</th>
<th>$x_6(t)$</th>
<th>$x_7(t)$</th>
<th>$x_8(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1.0</td>
<td>1.5</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
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<td>2.0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.2</td>
<td>0.3</td>
<td>0.4</td>
<td>0.5</td>
<td></td>
</tr>
</tbody>
</table>
Figure 10
Addiction of speed $v$ as a function of time

Figure 11
Phase portrait in coordinates $v(t) - x(t)$
3. Duboshynsky's pendulum

Let's consider the Duboshynsky pendulum (Fig. 12) - a mechanical pendulum that performs non-extinguishing quasi-natural oscillations due to interaction with a high-frequency variable magnetic field [3]. This effect was discovered by the brothers Danylo and Yakov Duboshynskyi in 1968-1969.

The Duboshynsky pendulum consists of two interacting parts:
- a mechanical pendulum with its own low frequency, with a small permanent magnet attached to its lower end;
- a stationary electromagnet located under the equilibrium point of the pendulum trajectory and powered by alternating current with a frequency from tens to thousands of hertz.

The permanent magnet at the end of the pendulum interacts with the magnetic field of the solenoid only on a limited part of the trajectory of the pendulum above the solenoid. This spatial heterogeneity of the interaction allows the pendulum to regulate its exchange of energy with the magnetic field. The decaying motion of the pendulum, initially released from any position, can turn into a steady, close to periodic...
one. With this movement, the pendulum in one or several periods of oscillation extracts a portion of energy from the interaction with the electromagnet, exactly compensating for frictional losses during the same time. Stability of oscillations is supported by self-adjustment of the phase relationship between the pendulum and the high-frequency field.

The amplitude of the established oscillations takes one stationary value from a discrete set of possible values for a given frequency of powering the electromagnet. Quantized amplitudes practically do not depend on the strength of the alternating current feeding the electromagnet. At the same time, the amplitudes are very sensitive to changes in the frequency of this current. The higher this frequency, the greater the number of quantized amplitudes that the pendulum can realize.

The equation of oscillations of the pendulum in question at fairly small deviations from the equilibrium position has the form

\[ \ddot{x} + 2\delta \dot{x} + \omega_0^2 (1 - \gamma x^2)x = f(x, t), \]

(4)

where \( f(x, t) \) – the interaction force between the pendulum and the electromagnet.

In equation (4), the linear and cubic terms in the turning force are preserved. Accounting for higher-order nonlinear terms does not bring anything fundamentally new to the results.

For simplicity, let us assume that the interaction force has the form

\[ f(x, t) = A \cos \omega t \]  

при  \(|x| \leq b\),

\[ = 0 \]  

при  \(|x| > b\),

that is

\[ f(x, t) = \mathcal{G}(b + x) \mathcal{G}(b - x) A \cos \omega t, \]

where \( \mathcal{G}(z) \) – Heaviside function.
We will look at such modes of pendulum swinging, for which the amplitude is much larger than the interval of mutual modality $b$.

For the butt, a drop-down with advanced parameters was looked at: $a = 6$, $b = 0.4$, $\delta = 0.3$, $\gamma = 0.2$, $\omega_0 = 1.5$, $\omega = 4$.

![Figure 13](image)

**Figure 13**
Dependency coordinates $x$ from time

![Figure 14](image)

**Figure 14**
Addiction of speed $v$ as a function of time
4. Electrostatic oscillator

Let's take a look at the dynamics of the electrostatic pendulum (Fig. 16) with the balancing of the friction forces. An electrically charged bag serves as a vantage in a mathematical pendulum. Mathematical electrostatic pendulum together with a wire grounded plate establishes a stable oscillator.
Equal to the swing of a physical pendulum: \( I\ddot{\alpha} = \sum M_e \), where \( I \) - the moment of inertia of the pendulum, but the axis of wrapping, \( M_e \) - the moments of the outer forces, which are blowing on a new one. At our vipadu you can record \( I\ddot{\alpha} = F_b \cos \alpha - mg \sin \alpha - \gamma \dot{\alpha} \). \( (5) \)

The last term in the right part of (5) is the moment of the friction force, proportional to the angular velocity. The Coulomb force acting on the ball from the side of its image in the conducting plate:

\[
F = \frac{kq^2}{4(L-b \sin \alpha)^2} = \frac{kq^2}{4b^2(\bar{L} - \sin \alpha)^2}, \quad \bar{L} = \frac{L}{b}.
\]

(Then we omit the tilde sign). Let's reduce the equation of motion to a dimensionless form by dividing both its parts by \( mg \), and the unit of measurement of time will be chosen so that the coefficient at \( \ddot{\alpha} \) was equal to one: \( [t] = \sqrt{\frac{mg}{I}} \). In its final form, the equation of motion of the pendulum will be as follows:

\[
\ddot{\alpha} + \beta \dot{\alpha} + \sin \alpha - \frac{Q^2 \cos \alpha}{(L - \sin \alpha)^2} = 0,
\]

\[
Q^2 = \frac{kq^2}{4mgab}, \quad \beta = \frac{\gamma}{\sqrt{Igma}}.
\]

(6)

Let's find the numerical solution of equation (6). To do this, we will reduce this differential equation of the second order to a system of differential equations of the first order. Let's introduce the notation \( y_1 = \alpha, \quad y_2 = \dot{\alpha} \), then equation (6) will be equivalent to the system

\[
\begin{align*}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= \beta y_2 - \sin y_1 + \frac{Q^2 \cos y_1}{(1 - \sin y_1)^2}
\end{align*}
\]
As initial conditions, we will take $\alpha(0) = 0$, $\dot{\alpha}(0) = 0$.

The graph of the dependence of the angle of deviation of the pendulum $\alpha$ at the moment of time $t$ on the equilibrium position is shown in Fig. 17...19.

![Figure 17](image1)

**Figure 17**
Dependence of the deviation angle $\alpha$ on time $t$ for different values of the parameter $Q(Q=0,3,6,8)$

![Figure 18](image2)

**Figure 18**
Dependence of the deviation angle $\alpha$ on time $t$ for different values of the parameter $\beta(\beta=0.2,0.3,0.4,0.5)$
From fig. 17, it can be seen that at $Q=0$, as $t$ increases, the oscillations decrease and the value of the deviation angle $\alpha$ goes to zero. If $Q$ is different from zero, then as $t$ increases, the oscillation amplitude decreases, but the value of the deviation angle $\alpha$ approaches a constant value that increases with $Q$. As $Q$ increases, the oscillation period also decreases.

As the parameter $\beta$ increases, the amplitude of oscillations decreases, but the period of oscillations and the value of the constant to which the deviation angle $\alpha$ tends as $t$ increases do not change, as can be seen from Fig. 18.

In fig. 19 shows the dependence of the angle of deviation $\alpha$ on time $t$ for different values of $L$, from which it can be seen that an increase in the parameter $L$ leads to a decrease in the amplitude of oscillations and the constant value to which the angle of deviation $\alpha$ goes as time $t$ increases.

**CONCLUSIONS**

The application of new design technologies, based on the use of mathematical modeling methods and computer technology, allows to ensure high quality and fast execution of design works. Mathematical modeling during design in most cases makes it possible to abandon physical modeling, significantly
reduce the amount of tests and finishing works, and ensure the creation of technical objects with high efficiency and quality indicators. In this case, a mathematical model becomes one of the main components of the design system.

References: